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## Coseparable coalgebras and coextensions of coderivations

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## COSEPARABLE COALGEBRAS AND COEXTENSIONS OF CODERIVATIONS

Dedicated to Professor Yoshikazu Nakai on his sixtieth birthday

ATSUSHI NAKAJIMA

Throughout the present paper,  $k$  will be a fixed field. All vector spaces are  $k$ -vector spaces and linear maps are  $k$ -linear. Unadorned  $\otimes$  means  $\otimes_k$ . As for notations and terminologies used here, we follow [4] and [5].

Let  $A$  and  $C$  be coalgebras, and  $\phi: A \rightarrow C$  a coalgebra map. If  $M$  is an  $A$ -comodule, then  $M$  is a  $C$ -comodule via  $\phi$ . Let  $M$  be an  $A$ - $A$ -bicomodule. A  $C$ -comodule map  $f: M \rightarrow A$  is called a *C-coderivation* if

$$\Delta_A f = (1 \square f) \rho^- + (f \square 1) \rho^+: M \rightarrow A \square_C A,$$

where  $\rho^-$  (resp.  $\rho^+$ ) is the left (resp. right)  $A$ -comodule structure map of  $M$ . A  $C$ -coderivation  $f$  is an *inner C-coderivation* if there exists a  $C$ -comodule map  $\tilde{r}: M \rightarrow C$  such that

$$f = (1 \square \epsilon_C \tilde{r}) \rho^- - (\epsilon_C \tilde{r} \square 1) \rho^+.$$

This is a generalization of the notion of a  $k$ -coderivation in the sense of Doi [2]. A  $C$ -coderivation  $\tau: M \rightarrow A$  is called a *coextension* of a  $k$ -coderivation  $\delta: M \rightarrow C$  if  $\phi\tau = \delta$ .

In what follows, we assume always that  $C$  is a cocommutative coalgebra,  $A$  is a  $C$ -coalgebra (i. e., a coalgebra over  $C$  via  $\phi$  ([4, p. 127])), and that  $M$  is an  $A$ - $A$ -bicomodule.

One of the purposes of this paper is to extend Doi's theorem on coseparable coalgebras [2, Th. 3] to coseparable  $C$ -coalgebras. We prove also the following: If  $A$  is a  $C$ -injective coalgebra and if  $H^2(N, A) = 0$  for all  $A$ - $A$ -bicomodules  $N$  (in the sense of Jonah [3, §4]), then for any  $k$ -coderivation  $\delta: M \rightarrow C$ , there exists a  $C$ -coderivation  $\tau: M \rightarrow A$  which is a coextension of  $\delta$ . Note that if  $A$  is a coseparable coalgebra, then  $H^2(N, A) = 0$ .

**1. Coseparable coalgebras.** In this section, we extend Doi's theorem on coseparable coalgebras [2, Th. 3] to our coseparable  $C$ -coalgebras. First, we consider the following exact sequence of  $A$ - $A$ -bicomodules

$$(1.1) \quad 0 \rightarrow A \xrightarrow{\Delta_A} A \square_C A \xrightarrow{\omega} (A \square_C A) / \Delta_A(A) \rightarrow 0$$

where  $\omega$  is the canonical linear map and the  $A$ - $A$ -bicomodule structure of  $(A \square_{\sigma} A) / \Delta_A(A)$  is defined naturally. We set  $L = (A \square_{\sigma} A) / \Delta_A(A)$  and  $a \circ b = \omega(a \square b)$ .

**Lemma 1.1** (cf. [2, § 3]). *A linear map  $\lambda: L \longrightarrow A$  defined by*

$$\lambda(a \circ b) = a \varepsilon_C \phi(b) - \varepsilon_C \phi(a) b$$

*is a  $C$ -coderivation.*

*Proof.* By the definition of  $C$ -coalgebras and the  $A$ - $A$ -bicomodule structure of  $L$ , we obtain  $\Delta_A \lambda = (1 \square \lambda) \rho_L^- + (\lambda \square 1) \rho_L^+$ , where  $\rho_L^-$  (resp.  $\rho_L^+$ ) is the right (resp. left)  $A$ -comodule structure map of  $L$ . It remains to show that  $\lambda$  is a left  $C$ -comodule map. We have

$$(\phi \otimes 1) \Delta_A \lambda(a \circ b) = \sum_{(a)} \phi(a_{(1)}) \otimes a_{(2)} \varepsilon_C \phi(b) - \sum_{(b)} \varepsilon_C \phi(a) \phi(b_{(1)}) \otimes b_{(2)}$$

and

$$\begin{aligned} (1 \otimes \lambda) (\phi \otimes 1) \rho_L^-(a \circ b) &= (1 \otimes \lambda) (\sum_{(a)} \phi(a_{(1)}) \otimes a_{(2)} \circ b) \\ &= \sum_{(a)} \phi(a_{(1)}) \otimes a_{(2)} \varepsilon_C \phi(b) - \phi(a) \otimes b. \end{aligned}$$

Since  $a \circ b$  is in  $L$ , we have

$$\sum_{(a)} a_{(1)} \otimes \phi(a_{(2)}) \otimes b = \sum_{(b)} a \otimes \phi(b_{(1)}) \otimes b_{(2)},$$

and therefore

$$\sum_{(b)} \varepsilon_C \phi(a) \phi(b_{(1)}) \otimes b_{(2)} = \sum_{(a)} \varepsilon_C \phi(a_{(1)}) \otimes \phi(a_{(2)}) \otimes b = \phi(a) \otimes b.$$

Hence  $(\phi \otimes 1) \Delta_A \lambda = (1 \otimes \lambda) (\phi \otimes 1) \rho_L^-$ , which shows that  $\lambda$  is a left  $C$ -comodule map. By the cocommutativity of  $C$ ,  $\lambda$  is a  $C$ -comodule map.

**Theorem 1.2.** *Let  $A$  be a  $C$ -coalgebra. Then the following conditions are equivalent:*

- (a)  *$A$  is a coseparable  $C$ -coalgebra.*
- (b) *For any  $A$ - $A$ -bicomodule  $M$ , every  $C$ -coderivation from  $M$  to  $A$  is an inner  $C$ -coderivation.*

*Proof.* (a)  $\implies$  (b). Since  $A$  is a coseparable  $C$ -coalgebra, there exists a linear map  $\tau: A \square_{\sigma} A \longrightarrow A$  such that  $\tau \Delta_A = 1$  and  $\Delta_A \tau = (1 \otimes \tau)(\Delta_A \square 1) = (\tau \otimes 1)(1 \square \Delta_A)$ . Hence we have

$$\tau = (1 \otimes \varepsilon_A \tau)(\Delta_A \tau \square 1) = (\varepsilon_A \otimes 1)(1 \square \Delta_A).$$

Let  $f: M \rightarrow A$  be an arbitrary  $C$ -coderivation. Setting  $h = \phi_\tau(1 \square f)\rho^-: M \rightarrow C$ , we can easily see that  $h$  is a  $C$ -comodule map. By the property of  $\tau$ , we have

$$(1 \square \varepsilon_C h)\rho^- = (1 \square \varepsilon_C \phi) \Delta_A \tau(1 \square f)\rho^- = \tau(1 \square f)\rho^-.$$

Since  $f = \tau \Delta_A f = \tau(1 \square f)\rho^- + \tau(f \square 1)\rho^+$  and  $\varepsilon_C \phi f = \varepsilon_A f = 0$ , we obtain that

$$\begin{aligned} (\varepsilon_C h \square 1)\rho^+ &= (\varepsilon_C \phi \square 1)((f - \tau(f \square 1)\rho^+) \square 1)\rho^+ = -(\varepsilon_C \phi \square 1) \Delta_A \tau(f \square 1)\rho^+ \\ &= -\tau(f \square 1)\rho^+. \end{aligned}$$

Therefore  $f = (1 \square \varepsilon_C h)\rho^- - (\varepsilon_C h \square 1)\rho^+$ , which shows that  $f$  is an inner  $C$ -coderivation.

(b)  $\Rightarrow$  (a). This can be proved by the same way as in the proof (iv)  $\Rightarrow$  (i) of [2, Th. 3]. For the sake of completeness, we give the proof. By assumption, there exists a  $C$ -comodule map  $\gamma: L \rightarrow C$  such that  $\lambda = (1 \square \varepsilon_C \gamma)\rho_L^- - (\varepsilon_C \gamma \square 1)\rho_L^+$ . Define  $\xi: L \rightarrow A \square {}_C A$  by  $\xi = (1 \square \varepsilon_C \square 1)(1 \square \gamma \square 1)(\rho_L^- \square 1)\rho_L^+$ . Then it is easy to see that  $\xi$  is an  $A$ - $A$ -bicomodule map and

$$\begin{aligned} \xi &= (1 \square \varepsilon_C \square 1)((\lambda \square 1 + (\gamma \square 1)\rho_L^+ \square 1)\rho_L^+ \\ &= (1 \square \varepsilon_C \square 1)((\lambda \square 1)\rho_L^+ + (1 \square \Delta_A)(\gamma \square 1)\rho_L^+) = (\lambda \square 1)\rho_L^+ + \Delta_A(\gamma \square 1)\rho_L^+. \end{aligned}$$

By  $\omega \Delta_A = 0$ , we have  $\omega \xi = \omega(\lambda \square 1)\rho_L^+$ . Finally, we shall show that  $\omega(\lambda \square 1)\rho_L^+ = 1$ . If  $a \circ b$  is in  $L$ , then

$$\begin{aligned} \omega(\lambda \square 1)\rho_L^+(a \circ b) &= \omega((\sum_{(b)} a \varepsilon_C \phi(b_{(1)}) - \varepsilon_C \phi(a) b_{(1)}) \otimes b_{(2)}) \\ &= a \circ b - \omega(\sum_{(b)} \varepsilon_C \phi(a) b_{(1)} \otimes b_{(2)}) = a \circ b. \end{aligned}$$

Thus the sequence (1.1) is split as  $A$ - $A$ -bicomodule.

**2. Coextensions of coderivations.** Let  $B$  be the direct sum of  $A$  and  $M$  as a vector space.

In [3], Jonah shows the following: Let  $\Delta_B: B \rightarrow B \otimes B$  and  $\varepsilon_B: B \rightarrow k$  be linear maps defined respectively by

$$\Delta_B = \begin{pmatrix} \Delta_A & 0 \\ 0 & \rho^- \\ 0 & \rho^+ \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \varepsilon_B = \begin{pmatrix} \varepsilon_A \\ 0 \end{pmatrix}.$$

Then  $(B, \Delta_B, \varepsilon_B)$  is a coalgebra.

Now, let  $\delta: M \rightarrow C$  be a  $k$ -coderivation, and let  $\rho_B: B \rightarrow C \otimes B$

be a linear map defined by

$$\rho_B = \begin{pmatrix} (\phi \otimes 1) \Delta_A & (\delta \otimes 1) \rho^+ \\ 0 & (\phi \otimes 1) \rho^- \end{pmatrix}.$$

We shall show that  $(B, \rho_B)$  is a left  $C$ -comodule. Since  $\delta$  is a  $k$ -coderivation and  $M$  is a  $C$ -comodule, we have  $\Delta_C \delta = (1 \otimes \delta)(\phi \otimes 1) \rho^- + (\delta \otimes 1)(1 \otimes \phi) \rho^+$ , and so

$$(2.1) \quad (\Delta_C \delta \otimes 1) \rho^+ = (\phi \otimes \delta \otimes 1)(1 \otimes \rho^-) \rho^- + (\delta \otimes \phi \otimes 1)(1 \otimes \Delta_A) \rho^+.$$

Moreover it is easy to see that

$$(\Delta_C \otimes 1) \rho_B = \begin{pmatrix} (\Delta_C \phi \otimes 1) \Delta_A & (\Delta_C \otimes 1)(\delta \otimes 1) \rho^+ \\ 0 & (\Delta_C \otimes 1)(\phi \otimes 1) \rho^- \end{pmatrix}$$

and

$$(1 \otimes \rho_B) \rho_B = \begin{pmatrix} (\phi \otimes \phi \otimes 1)(1 \otimes \Delta_A) \Delta_A & (\delta \otimes \phi \otimes 1)(1 \otimes \Delta_A) \rho^+ + (\phi \otimes \delta \otimes 1)(1 \otimes \rho^+) \rho^- \\ 0 & (\phi \otimes \phi \otimes 1)(1 \otimes \rho^-) \rho^- \end{pmatrix}$$

Then by (2.1), we have  $(\Delta_B \otimes 1) \rho_C = (1 \otimes \rho_B) \rho_B$  and

$$(\epsilon_C \otimes 1) \rho_B = \begin{pmatrix} (\epsilon_C \otimes 1)(\phi \otimes 1) \Delta_A & (\epsilon_C \otimes 1)(\delta \otimes 1) \rho^+ \\ 0 & (\epsilon_C \otimes 1)(\phi \otimes 1) \rho^- \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus  $B$  is a left  $C$ -comodule.

Finally, by making use of the structure of  $A \oplus M$  mentioned above, we shall prove the following

**Theorem 2.1.** *Let  $A$  be a  $C$ -coalgebra, and let  $\delta: M \rightarrow C$  be a  $k$ -coderivation. If  $A$  is an injective  $C$ -comodule and if  $H^2(N, A) = 0$  for all  $A$ - $A$ -bicomodules  $N$ , then there exists a  $C$ -coderivation  $\tilde{\delta}: M \rightarrow A$  which is a coextension of  $\delta$ .*

*Proof.* As is claimed above,  $B = A \oplus M$  is a coalgebra and a  $C$ -comodule. Since the canonical projection  $B \rightarrow A$  is a coalgebra map,  $B$  is a  $C$ -coalgebra. Consider the exact sequence of  $C$ -comodules

$$(2.2) \quad 0 \rightarrow A \xrightarrow{i} B \xrightarrow{p} M \rightarrow 0$$

where  $i$  is the canonical injection and  $p$  is the canonical projection. Then (2.2) is a singular coalgebra extension ([3, § 4]). By the  $C$ -injectivity of  $A$ , (2.2) is split as  $C$ -comodule. Hence by [3, Th. 4.10], there exists a  $C$ -coalgebra map  $\tilde{\delta}: B \rightarrow A$  such that  $\tilde{\delta}i = 1$ . Identifying  $m$  in  $M$  with  $\begin{pmatrix} 0 \\ m \end{pmatrix}$  in  $B$ ,  $\Delta_A \tilde{\delta} = (\tilde{\delta} \otimes \tilde{\delta}) \Delta_B$  implies

$$(2.3) \quad \Delta_A \tilde{\delta}(m) = (1 \otimes \tilde{\delta})\rho^-(m) + (\tilde{\delta} \otimes 1)\rho^+(m).$$

Thus  $\tilde{\delta}$  is a  $C$ -coderivation. Moreover, since  $\tilde{\delta}$  is a  $C$ -comodule map we have

$$(\tilde{\phi} \otimes 1)\Delta_A \tilde{\delta} = (1 \otimes \tilde{\delta}) \begin{pmatrix} (\phi \otimes 1)\Delta_A & (\delta \otimes 1)\rho^+ \\ 0 & (\phi \otimes 1)\rho^- \end{pmatrix}$$

and so by (2.3), we obtain

$$((\phi \otimes \tilde{\delta})\rho^+ + (\phi \tilde{\delta} \otimes 1)\rho^-)(m) = ((\delta \otimes 1)\rho^+ + (\phi \otimes \tilde{\delta})\rho^-)(m).$$

Therefore  $(\phi \tilde{\delta} \otimes 1)\rho^+ = (\delta \otimes 1)\rho^+$  on  $M$ . This shows that  $\phi \tilde{\delta} = \delta$ .

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